Integrals of Motion for Planar Multi-Body Formations with Internal Forces

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Interest in the use of non-contacting forces between spacecraft has prompted many studies of the dynamics of such formations. While the introduction of such a force potentially complicates the analysis of these systems, integrals of motion still exist for idealized cases. These integrals not only define relationships between the states at two different times, but also provide a means to describe the error introduced through application of simplifying assumptions to the formation dynamics. This paper develops expressions of two integrals of motion for a planar, two-vehicle formation, and examines their evolution under several assumptions pertaining to the motion of the formation center of mass.

Nomenclature

A, B	= Constraint matricies
a_1, a_2	= Semimajor axes of spacecraft
C_E	= Total mechanical energy
C_H	= Total angular momentum magnitude
e_1, e_2	= Eccentricities of spacecraft
$\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_c$, $\mathbf{e}_\rho$, $\mathbf{e}_\gamma$, $\mathbf{e}_\theta$	= Basis vectors
$\mathbf{h}_1$, $\mathbf{h}_2$	= Angular momentum vectors of individual spacecraft
$\mathcal{L}$	= Lagrangian of dynamics system
m_1, m_2, M	= Masses of individual spacecraft, sum of masses
q	= Generalized coordinates

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$\tilde{q}$	= Constrained generalized coordinates
$\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_c$, $\boldsymbol{\rho}$	= Position vectors
r_1, r_2, r_c, ρ	= Magnitudes of position vectors
$\mathcal{R}$	= Interaction potential energy
$\mathcal{T}$, $\mathcal{V}$	= Total kinetic, potential energies
x	= System state of original system
$\bar{x}$	= System state of simplified system
γ, θ	= Angles describing orientation of formation
λ_1, λ_2	= Lagrange multipliers
μ	= Gravitational parameter

I. Introduction

Studying the relative motion of multiple spacecraft in orbit about a common central body has proven useful for such tasks as rendezvous and docking, in-orbit inspection, and mission-enabling formation flight. An examination of the effects of independent control inputs acting on individual spacecraft informs mission planning and controller design by predicting the evolving Keplerian orbits of the vehicles through a maneuver. This classic approach of utilizing external forces as control inputs allows for the arbitrary reconfiguration of the multi-body system. Augmenting the dynamics of a spacecraft formation with an action-at-a-distance force internal to the multi-body system presents a unique opportunity by enabling control of the relative motion of the spacecraft while maintaining certain integrals of motion. This paper examines the dynamics of a planar, two-vehicle formation in orbit about a common central body with such an internal force acting between the spacecraft. Through this study, integrals of motion are identified and used to describe relationships bounding the system state trajectory and to evaluate a common assumption regarding the motion of the formation center of mass.

While the unperturbed motion of a spacecraft in a Keplerian orbit about a central body is well-known, analytical expressions of its state trajectory do not exist in the presence of arbitrary perturbations. At least two related approaches begin to address this problem. The first approach identifies conserved quantities relating the states, such as with the Jacobi Integral placing bounds on the trajectory of a point mass in the Circular Restricted Three-Body Problem.[1] The second approach linearizes the system's dynamics about an assumed center of mass trajectory, resulting in a set of first-order ordinary differential equations.[2] These simplifying assumptions can be framed as artificially enforcing system state behavior in contrast with the natural known behavior of the unmodified dynamic system. Both of these approaches reduce the total number of states required to represent the dynamics of

the system, simplifying controller design and potentially bounding the state trajectory. We address both of the above approaches through the development of a Lagrangian dynamics model. Through this method, integrals of the motion representing conserved quantities can quickly be identified and applied to the example system. In addition to reducing the number of states of the system, these conserved quantities of the original system also provide a means to evaluate the applicability of simplifying assumptions. Applying such an assumption regarding the motion of the system is equivalent to enforcing a constraint on its dynamics, and the resulting hypothetical constraint force potentially modifies the otherwise conserved integrals of motion of the original system. Several groups have investigated the specific topic of energy- and momentum-conserving integration schemes.[3–6] While these methods could be employed in the analysis of the two-vehicle system under discussion, this paper focuses on the interpretation of the relationship between the error introduced in the generalized coordinates and the conserved quantities under a common simplifying assumptions regarding the center of mass motion of the formation.

The paper begins by developing the initial mathematical model of the planar, two-vehicle system dynamics and identifies relevant integrals of motion in section II. Subsequently, section III examines how these integrals of motion translate into equations relating the osculating orbital elements of the individual vehicles. Section IV then examines some possible assumptions regarding the evolution of the system's state. The impact of a center-of-mass-motion constraint is then examined in terms the effect upon the integrals of motion derived from the original system. The paper then supports the theoretical development through a simulated example formation in section V.

II. Dynamics Model Description

Consider the following planar system describing two masses, m_1 and m_2, relative to a large central body. Their locations are described by $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively, and their center of mass is defined by the vector $\mathbf{r}_c$. The vector $\boldsymbol{\rho}$ points from m_1 to m_2. Mathematically, we can describe these vectors in terms of each other.

$$\begin{aligned} \mathbf{r}_c =& \frac{1}{M}\left(m_1\mathbf{r}_1 + m_2\mathbf{r}_2\right) = r_c\mathbf{e}_c & \boldsymbol{\rho} =& \mathbf{r}_2 - \mathbf{r}_1 = \rho\mathbf{e}_\rho \\ \mathbf{r}_1 =& \mathbf{r}_c - \frac{m_2}{M}\boldsymbol{\rho} & \mathbf{r}_2 =& \mathbf{r}_c + \frac{m_1}{M}\boldsymbol{\rho} \end{aligned} \tag{1}$$

From these equations, we can define the inertial time derivatives of $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of inertial time derivatives of $\mathbf{r}_c$ and $\boldsymbol{\rho}$.

$$\dot{\mathbf{r}}_1 = \dot{\mathbf{r}}_c - \frac{m_2}{M}\dot{\boldsymbol{\rho}} \qquad \dot{\mathbf{r}}_2 = \dot{\mathbf{r}}_c + \frac{m_1}{M}\dot{\boldsymbol{\rho}} \tag{2}$$

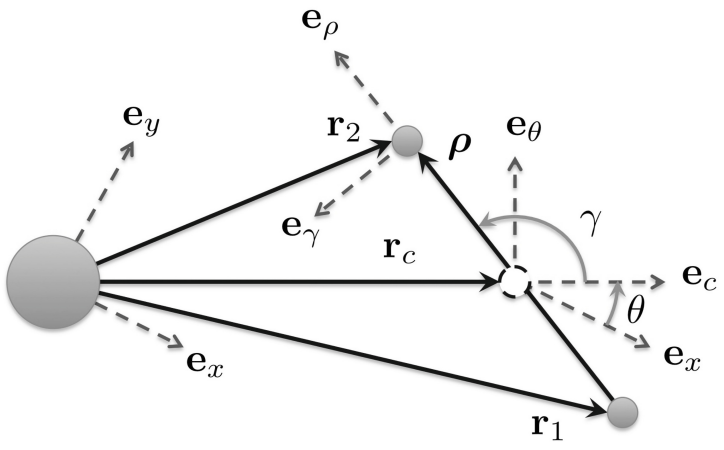

Figure 1. Layout of m_1 and m_2 relative to a central body.

Both masses are gravitationally attracted to the central body and experience a conservative force directed along the vector $\mathbf{e}_\rho$ pointing between the two masses. This force is derived from the potential function $\mathcal{R}$ and is assumed to only depend upon the relative separation distance ρ.

A. Deriving Equations of Motion

The Lagrangian $\mathcal{L}$ is formed from the kinetic and potential energy expressions, $\mathcal{T}$ and $\mathcal{V}$.

$$\mathcal{T} = \frac{1}{2}m_1\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 + \frac{1}{2}m_2\dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2 \tag{3}$$

$$\mathcal{V} = -\mu\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) + \mathcal{R} \tag{4}$$

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = \frac{1}{2}m_1\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 + \frac{1}{2}m_2\dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2 + \mu\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) - \mathcal{R} \tag{5}$$

The position vectors of Eq. (1) can be expressed in terms of scalar components multiplying basis vectors the depicted in Fig. 1.

$$\mathbf{r}_1 = \left(r_c - \frac{m_2}{M}\rho\cos\gamma\right)\mathbf{e}_c - \frac{m_2}{M}\rho\sin\gamma\mathbf{e}_\theta \qquad \mathbf{r}_2 = \left(r_c + \frac{m_1}{M}\rho\cos\gamma\right)\mathbf{e}_c + \frac{m_1}{M}\rho\sin\gamma\mathbf{e}_\theta \tag{6}$$

Similarly, we can write the inertial velocity vectors in terms of orthogonal basis vectors $\mathbf{e}_c$

and $\mathbf{e}_\theta$.

$$\begin{aligned}\dot{\mathbf{r}}_1 = &\left(\dot{r}_c - \frac{m_2}{M}\left(\dot{\rho}\cos\gamma - \rho\left(\dot{\theta}+\dot{\gamma}\right)\sin\gamma\right)\right)\mathbf{e}_c \\ &+\left(r_c\dot{\theta} - \frac{m_2}{M}\left(\dot{\rho}\cos\gamma + \rho\left(\dot{\theta}+\dot{\gamma}\right)\sin\gamma\right)\right)\mathbf{e}_\theta \end{aligned} \tag{7}$$

$$\begin{aligned}\dot{\mathbf{r}}_2 = &\left(\dot{r}_c + \frac{m_1}{M}\left(\dot{\rho}\cos\gamma - \rho\left(\dot{\theta}+\dot{\gamma}\right)\sin\gamma\right)\right)\mathbf{e}_c \\ &+\left(r_c\dot{\theta} + \frac{m_1}{M}\left(\dot{\rho}\cos\gamma + \rho\left(\dot{\theta}+\dot{\gamma}\right)\sin\gamma\right)\right)\mathbf{e}_\theta \end{aligned} \tag{8}$$

Consider the following choice of generalized coordinates:

$$q = \begin{bmatrix} \rho & \gamma & r_c & \theta \end{bmatrix}^T \tag{9}$$

We can define the lengths of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of q.

$$\begin{aligned} r_1 = &r_1(q) = (\mathbf{r}_1\cdot\mathbf{r}_1)^{1/2} = \left(q_3^2 - 2\frac{m_2}{M}q_1q_3\cos q_2 + \left(\frac{m_2}{M}q_1\right)^2\right)^{1/2} \\ r_2 = &r_2(q) = (\mathbf{r}_2\cdot\mathbf{r}_2)^{1/2} = \left(q_3^2 + 2\frac{m_1}{M}q_1q_3\cos q_2 + \left(\frac{m_1}{M}q_1\right)^2\right)^{1/2} \end{aligned} \tag{10}$$

The expressions for the energy terms and Lagrangian from Eqs. (3), (4), and (5) are then rewritten in terms of the generalized coordinates.

$$\mathcal{T}(q,\dot{q}) = \frac{1}{2}\left(M\left(\dot{q}_3^2 + q_3^2\dot{q}_4^2\right) + \frac{m_1m_2}{M}\left(\dot{q}_1^2 + q_1^2\left(\dot{q}_2+\dot{q}_4\right)^2\right)\right) \tag{11}$$

$$\mathcal{V}(q) = -\mu\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) + \mathcal{R}(q) \tag{12}$$

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}\left(M\left(\dot{q}_3^2 + q_3^2\dot{q}_4^2\right) + \frac{m_1m_2}{M}\left(\dot{q}_1^2 + q_1^2\left(\dot{q}_2+\dot{q}_4\right)^2\right)\right) + \mu\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) - \mathcal{R}(q) \tag{13}$$

The equation of motion describing the dynamics of the generalized coordinate q_j for a system acted upon only by conservative forces is given by the Lagrange's equation:

$$\frac{d}{dt}\frac{\partial}{\partial\dot{q}_j}\mathcal{L} - \frac{\partial}{\partial q_j}\mathcal{L} = 0 \tag{14}$$

Applying Eq. (14) to Eq. (13) provides the set of second-order differential equations de-

scribing the dynamics of the system.

$$\ddot{q}_1 - q_1\left(\dot{q}_2+\dot{q}_4\right)^2 + \frac{\mu}{r_1^3}\left(\frac{m_2}{M}q_1 - q_3\cos q_2\right) + \frac{\mu}{r_2^3}\left(\frac{m_1}{M}q_1 + q_3\cos q_2\right) + \frac{M}{m_1 m_2}\frac{\partial}{\partial q_1}\mathcal{R} = 0 \quad (15)$$

$$q_1^2\left(\ddot{q}_2+\ddot{q}_4\right) + 2q_1\dot{q}_1\left(\dot{q}_2+\dot{q}_4\right) + \mu\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)q_1 q_3 \sin q_2 = 0 \quad (16)$$

$$\ddot{q}_3 - q_3\dot{q}_4^2 + \mu\frac{m_1 m_2}{M}\left(\frac{1}{r_1^3}\left(\frac{M}{m_2}q_3 - q_1\cos q_2\right) + \frac{1}{r_2^3}\left(\frac{M}{m_1}q_3 + q_1\cos q_2\right)\right) = 0 \quad (17)$$

$$q_1^2\left(\ddot{q}_2+\ddot{q}_4\right) + 2q_1\dot{q}_1\left(\dot{q}_2+\dot{q}_4\right) + \frac{M^2}{m_1 m_2}\left(q_3\ddot{q}_4 + 2q_3\dot{q}_3\dot{q}_4\right) = 0 \quad (18)$$

B. Integrals of Motion

Deriving Eqs. (15), (16), (17), and (18) by a Lagrangian approach allows for quick identification of two integrals of motion. One of these integrals corresponds to the conservation of mechanical energy. Because the potential energy, $\mathcal{V}$, only depends upon the positions of m_1 and m_2, its time derivative can be written in terms of inertial velocities multiplying partials of $\mathcal{V}$ with respect to the position vectors.

$$\frac{d}{dt}\mathcal{V} = \dot{\mathbf{r}}_1 \cdot \frac{\partial \mathcal{V}}{\partial \mathbf{r}_1} + \dot{\mathbf{r}}_2 \cdot \frac{\partial \mathcal{V}}{\partial \mathbf{r}_2} \quad (19)$$

The forces resulting from the potential function are also related to these partial derivatives of $\mathcal{V}$.

$$m_1\ddot{\mathbf{r}}_1 = -\frac{\partial \mathcal{V}}{\partial \mathbf{r}_1} \qquad m_2\ddot{\mathbf{r}}_2 = -\frac{\partial \mathcal{V}}{\partial \mathbf{r}_2} \quad (20)$$

Incorporating these results into the expression time derivative of $\mathcal{T}$ reveals a relation between the derivatives of the kinetic and potential energy.

$$\begin{aligned}\frac{d}{dt}\mathcal{T} =& m_1\dot{\mathbf{r}}_1\cdot\ddot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2\cdot\ddot{\mathbf{r}}_2 \\ =& -\frac{d\mathbf{r}_1}{dt}\cdot\frac{\partial\mathcal{V}}{\partial\mathbf{r}_1} - \frac{d\mathbf{r}_2}{dt}\cdot\frac{\partial\mathcal{V}}{\partial\mathbf{r}_2} \\ =& -\frac{d\mathcal{V}}{dt}\end{aligned} \quad (21)$$

Summing the expressions for the time derivative of the kinetic and potential energies shows that the total energy, C_E, is conserved.

$$\frac{d}{dt}\left(\mathcal{T}+\mathcal{V}\right) = \frac{d}{dt}C_E = 0 \quad (22)$$

Equation (22) defines C_E as a constant that can be related at different times in the state trajectory time history.

$$C_E = \frac{1}{2}\left(M\left(\dot{q}_3^2 + q_3^2\dot{q}_4^2\right) + \frac{m_1 m_2}{M}\left(\dot{q}_1^2 + q_1^2\left(\dot{q}_2 + \dot{q}_4\right)^2\right)\right) - \mu\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) + \mathcal{R}\left(q\right) \tag{23}$$

The lack of q_4 in Eq. (13) suggests another integral of motion. Writing Eq. (14) for q_4 reveals this second time-independent quantity.

$$\frac{d}{dt}\frac{\partial}{\partial\dot{q}_4}\mathcal{L} - \frac{\partial}{\partial q_4}\mathcal{L} = \frac{d}{dt}\frac{\partial}{\partial\dot{q}_4}\mathcal{L} = 0 \tag{24}$$

This equation corresponds to the conservation of total angular momentum of the system.

$$C_H = \frac{\partial}{\partial\dot{q}_4}\mathcal{L} = Mq_3^2\dot{q}_4 + \frac{m_1 m_2}{M}q_1^2\left(\dot{q}_2 + \dot{q}_4\right) \tag{25}$$

While neither Eq. (23) nor Eq. (25) should come as a surprise, the process of defining these quantities in terms of the generalized coordinates will prove useful in the examination of the nonlinear system described by Eqs. (15), (16), (17), and (18).

III. Bounding System Output

By defining time-independent relationships between the generalized coordinates, C_E and C_H potentially provide a means to glean information about the system at a later time without the need to numerically or analytically integrate the equations of motion. Consider, for example, the evolution of the semimajor axes of m_1 and m_2 through the application of the forces associated with $\mathcal{R}$. Assuming a mission designer has control over the input of this conservative force acting on the two masses, an orbit-change maneuver affecting both vehicles would result from applying the force for only a limited duration. The semimajor axes of m_1 and m_2 are defined by the mechanical energy associated with the inertial velocity and gravitational potential of mass individually in the corresponding two-body problem.[7]

$$m_i\left(\frac{\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i}{2} - \frac{\mu}{r_i}\right) = -\frac{\mu m_i}{2a_i} \tag{26}$$

The total mechanical energy of the system can then be rearranged to include terms associated with the osculating semimajor axis a_i.

$$\begin{aligned} C_E =& m_1 \left(\frac{\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1}{2} - \frac{\mu}{r_1} \right) + m_2 \left(\frac{\dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2}{2} - \frac{\mu}{r_2} \right) + \mathcal{R} \\ =& -\frac{\mu}{2} \left(\frac{m_1}{a_1} + \frac{m_2}{a_2} \right) + \mathcal{R} \end{aligned} \tag{27}$$

As C_E does not vary with time, this equation can be used to relate changes in the interaction potential, $\mathcal{R}$, to changes in the semimajor axes of m_1 and m_2.

$$-\frac{\mu}{2} \left(\frac{m_1}{a_{1,0}} + \frac{m_2}{a_{2,0}} \right) + \mathcal{R}_0 = -\frac{\mu}{2} \left(\frac{m_1}{a_{1,f}} + \frac{m_2}{a_{2,f}} \right) + \mathcal{R}_f \tag{28}$$

Another way to interpret Eq. (27) is to recognize that the the combination of a_1, a_2, and R need to exist on a surface specified by the constants C_E, μ, m_1, and m_2. Figure 2 provides an example surface generated by the simulation parameters from Table 1.

A similar derivation relates the total angular momentum of the system to the semimajor axes and orbital eccentricities of m_1 and m_2 under the assumptions of two-body motion. The eccentricity of each mass is defined by an equation depending upon its individual angular momentum and semimajor axis.[7]

$$e_i = \sqrt{1 - \frac{\mathbf{h}_i \cdot \mathbf{h}_i}{\mu m_i^2 a_i}} \tag{29}$$

To total angular momentum of the system is also defined by summing up the contributions from the individual vehicles. This allows for the expression of the constant angular momentum magnitude, C_H, in terms of semimajor axis and eccentricity parameters.

$$C_H = \sqrt{\mu a_1 m_1^2 \left(1 - e_1^2\right)} + \sqrt{\mu a_2 m_2^2 \left(1 - e_2^2\right)} \tag{30}$$

Similar to the relation between Eq. (27) and Fig. 2, Eq. (30) describes a hyperdimensional surface parameterized by a_1, a_2, e_1, and e_2 for a given set of constants C_H, μ, m_1, and m_2.

The conservation of energy and angular momentum can be expressed in either generalized coordinates, as in Eqs. (23) and (25), or in terms of a set of outputs, such as the selection of osculating orbital elements in Eqs. (27) and (30). In either case, these equations provide a means to analytically bound the motion of the system, as any combination of states or outputs violating these conservation laws for a given set of initial conditions cannot be reached.

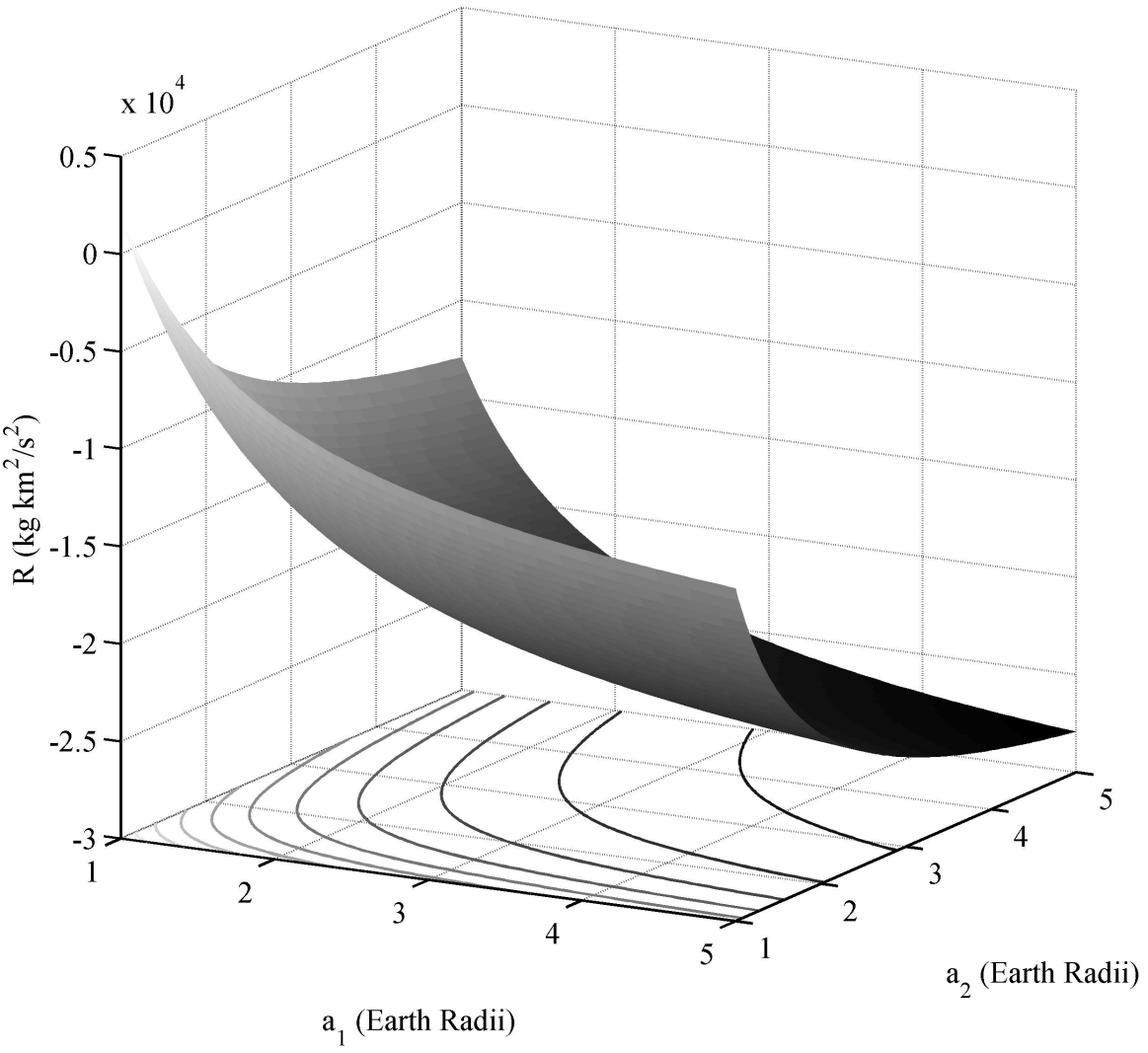

Figure 2. Surface and contour plot of interaction potential R in relation to semimajor axes a_1 and a_2 using the parameters from Table 1.

IV. Implications for Simplifying Assumptions

Integrals of motion also may be utilized when attempting to simplify the mathematical expression of the dynamics of a system. By defining conserved quantities, such as C_H for Eqs. (15), (16), (17), and (18), the time derivative of a particular state can be defined in terms of the remaining states and their time derivatives and can be substituted into the equations of motion, which effectively reduces the number of differential equations to be solved. Mathematically, a similar process occurs when applying constraints to the system dynamics. Defining specified states through constraint equations removes the need to solve for those states by other means. Specifying such constraints, however, generally does not preserve the conservation laws defining the integrals of motion derived for the unconstrained system.

A common set of simplifying assumptions for the relative motion of two spacecraft in orbit restricts the motion of the formation center of mass to follow a defined path, usually a Keplerian orbit.[8,9] This approach is similar to utilizing the Clohessy-Wiltshire equations for relative motion in near-circular orbits.[2] As a result, the relative motion of the two vehicles can be examined independently from the dynamics of the center of mass. The quality and applicability of such constraints could be evaluated by examining the evolution of the integrals of motion of the unconstrained system. Kim and Schaub formulate this problem in terms of orbital element differences and investigate angular momentum conservation for a variety of reference trajectories, including the enter of mass motion and the mass-averaged orbital elements.[10] As these quantities are not necessarily conserved in the constrained system, their variation is linked to the error between the constrained and unconstrained states.

The equations of motion of a constrained system are derived utilizing Lagrange multipliers.[11] When m constraints applied to a system with n generalized coordinates are linear in the constrained state derivatives, the constraint equations take the following form:

$$\sum_{j=1}^{n} A_{kj}\dot{q}_j + B_k = 0 \qquad\qquad k = 1, 2, ..., m \tag{31}$$

The equations of motion of the state q_j of this constrained system then follow the constrained form of Lagrange's equation.

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\mathcal{L} - \frac{\partial}{\partial q_j}\mathcal{L} = \sum_{k=1}^{m} \lambda_k A_{kj} \tag{32}$$

The λ_k values correspond to the constraint forces and torques applied to the system dynamics.[11]

A. Constrained Equations of Motion

Consider the same two-vehicle formation in orbit about a common central body as depicted in Fig. 1, but with the addition of two constraint equations defining the time history of the center of mass motion described by q_3 and q_4.

$$q_3 = \tilde{q}_3 \qquad\qquad q_4 = \tilde{q}_4 \tag{33}$$

As the $\tilde{q}_3$ and $\tilde{q}_4$ are assumed to be known functions of time, the time derivatives of q_3 and q_4 are also defined. Reformatting these two constraints results in a set of linear equations conforming to the format of Eq. (31).

$$A\dot{q} + B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dot{q} - \begin{bmatrix} \dot{\tilde{q}}_3 \\ \dot{\tilde{q}}_4 \end{bmatrix} = 0 \tag{34}$$

As the matrix A involves only the states q_3 and q_4, Eqs. (15), (16), and (34) define the dynamics of the constrained system. The λ_k values are defined through Eq. (32).

$$\lambda_1 = \ddot{\tilde{q}}_3 - \tilde{q}_3\dot{\tilde{q}}_4^2 + \mu\frac{m_1 m_2}{M}\left(\frac{1}{\tilde{r}_1^3}\left(\frac{M}{m_2}\tilde{q}_3 - q_1\cos q_2\right) + \frac{1}{\tilde{r}_2^3}\left(\frac{M}{m_1}\tilde{q}_3 + q_1\cos q_2\right)\right) \tag{35}$$

$$\lambda_2 = q_1^2\left(\ddot{q}_2 + \ddot{\tilde{q}}_4\right) + 2q_1\dot{q}_1\left(\dot{q}_2 + \dot{\tilde{q}}_4\right) + \frac{M^2}{m_1 m_2}\left(\tilde{q}_3\ddot{\tilde{q}}_4 + 2\tilde{q}_3\dot{\tilde{q}}_3\dot{\tilde{q}}_4\right) \tag{36}$$

In the case where the system naturally followed the constraints imposed by Eq. (34), λ_1 and λ_2 would be zero for all time. An examination of Eq. (36) with the definition of C_H given by Eq. (25) defines the time-dependence of the C_H integral of motion of the unconstrained system.

$$\frac{d}{dt}\frac{\partial}{\partial\dot{q}_4}\mathcal{L} - \frac{\partial}{\partial q_4}\mathcal{L} = \frac{d}{dt}C_H = \lambda_2 \tag{37}$$

Equation (37) demonstrates that for the constrained system and a general choice of $\tilde{q}_3$ and $\tilde{q}_4$ trajectories resulting in a non-zero λ_2, angular momentum is not conserved.

B. Constrained Equations of Motion Conserving C_H

A possible further constraint to be applied to the original unconstrained system would enforce the conservation of angular momentum. Requiring λ_2 to be zero effectively enforces this constraint. Substituting Eq. (34) into Eq. (25) and rearranging provides an expression of $\dot{q}_2$ in terms of only q_1 and known quantities. This differential equation and the constrained version of Eq. (15) define the state dynamics of the constrained system conserving angular

momentum.

$$\ddot{q}_1 - q_1\left(\dot{q}_2 + \dot{\tilde{q}}_4\right)^2 + \frac{\mu}{\tilde{r}_1^3}\left(\frac{m_2}{M}q_1 - \tilde{q}_3\cos q_2\right) + \frac{\mu}{\tilde{r}_2^3}\left(\frac{m_1}{M}q_1 + \tilde{q}_3\cos q_2\right) + \frac{M}{m_1 m_2}\frac{\partial}{\partial q_1}\mathcal{R} = 0 \quad (38)$$

$$\dot{q}_2 = \frac{M}{q_1^2 m_1 m_2}\left(C_H - M\tilde{q}_3^2\dot{\tilde{q}}_4\right) - \dot{\tilde{q}}_4 \quad (39)$$

While λ_2 is necessarily zero in this formulation, λ_1 is still allowed to be a non-zero value and can still function as a measure of the applicability of the constraining assumptions.

C. Integrals of Motion and Assumption Quality

Conservation laws, such as those defining C_E and C_H in Eqs. (23) and (25), are potentially useful for bounding the state or output error of a system under a set of simplifying assumptions. As demonstrated in the derivation of Eq. (37), these integrals of motion of the original system are not necessarily conserved in the simplified version. Additionally, the true error in these 'conserved' quantities is directly available for a given state trajectory by evaluating the quantity at a given time and subtracting the evaluation at the simulation initial conditions.

For a system conserving energy and angular momentum, this error equation takes a simple form. The true state, $x(t) = \begin{bmatrix} q^T & \dot{q}^T \end{bmatrix}^T$, conserves these values for all time, while the state under the simplified system, $\bar{x}(t) = \begin{bmatrix} \bar{q}^T & \dot{\bar{q}}^T \end{bmatrix}^T$, does not necessarily do so.

$$\begin{bmatrix} \Delta C_H(t) \\ \Delta C_E(t) \end{bmatrix} = \begin{bmatrix} C_H(\bar{x}(t)) \\ C_E(\bar{x}(t)) \end{bmatrix} - \begin{bmatrix} C_H(x(t)) \\ C_E(x(t)) \end{bmatrix} = \begin{bmatrix} C_H(\bar{x}(t)) - C_H(\bar{x}(t_0)) \\ C_E(\bar{x}(t)) - C_E(\bar{x}(t_0)) \end{bmatrix} \quad (40)$$

Equations (23) and (25) can be linearized about the state $\bar{x}$, effectively producing a linear relationship of an exactly known error in C_E and C_H to small approximate errors in the states of the simplified system.

$$\begin{bmatrix} \Delta C_H(t) \\ \Delta C_E(t) \end{bmatrix} \approx \begin{bmatrix} \frac{\partial}{\partial q}C_H & \frac{\partial}{\partial \dot{q}}C_H \\ \frac{\partial}{\partial q}C_E & \frac{\partial}{\partial \dot{q}}C_E \end{bmatrix}_{\bar{x}} \left[\bar{x}(t) - x(t)\right]^T = H(t)\,\Delta x(t) \quad (41)$$

V. Simulation Results

This section studies an example formation under an assortment of simplifying assumptions regarding the motion of the center of mass. The original system models a two-vehicle formation with a constant force, K, acting along the separation vector in a nearly circular orbit about the Earth with an altitude of 700 km. This particular forcing model simulates

the interaction due to a 100 W Photonic Laser Propulsion (PLP) system acting between the two vehicles.[12,13] The parameters and initial conditions of the system are given by Table 1.

Table 1. Simulation parameters

μ	3.986×10^5	km^3/s^2	K	0.223	N
m_1	600	kg	m_2	600	kg
ρ_0	100	km	$\dot{\rho}_0$	0	km/s
γ_0	$\pi/2$	rad	$\dot{\gamma}_0$	0	rad/s
$r_{c,0}$	6752	km	$\dot{r}_{c,0}$	0	km/s
θ_0	0	rad	$\dot{\theta}_0$	1.138×10^{-3}	rad/s

In addition to simulating the true dynamics of the system given by Eqs. (15), (16), (17), and (18), two sets of additional simulations integrate the equations of motion with particular constraints applied to the motion of the center of mass as described in subsection A of section IV. The first set assumes that the center of mass travels on a Keplerian orbit determined by its initial conditions. The second set assumes that the center of mass travels on a path that it would follow for the same system without the internal force acting between the two vehicles. The time histories of the motion of the center of mass for both of these constrained sets of systems are available *a priori*, as they respectively represent the true Keplerian motion of a single mass and the averaged Keplerian motion of two masses. Within each of these two sets of assumptions regarding the motion of the center of mass, a further constraint is optionally included to enforce conservation of angular momentum as described in subsection B of section IV.

Figure 3 depicts the error in the system states of the constrained system where the center of mass follows a Keplerian orbit. Similarly, Figure 4 depicts the error in the system states of the constrained system where the center of mass follows the weighted average of the Keplerian motion of the spacecraft without the PLP interaction. Each of these figures individually indicates that the state error of the constrained system is relatively independent of the decision to enforce conservation of angular momentum for the parameters and initial conditions described in Table 1. Comparison of the two figures likewise indicates that while both assumptions regarding the motion of the center of mass are similar in error magnitude for the relative position states, ρ and γ, the constrained system producing Fig. 4 results in lower state errors for the motion of the center of mass. Figure 5 interprets the state error produced by the simulation in terms of the error in the orbital parameters a_1, a_2, e_1, and e_2. While these four quantities do not alone allow for the full reconstruction of the system state, they do highlight significant differences in the error introduced by the center of mass motion assumptions. Utilizing the averaged natural orbital motion of the system in the absence of

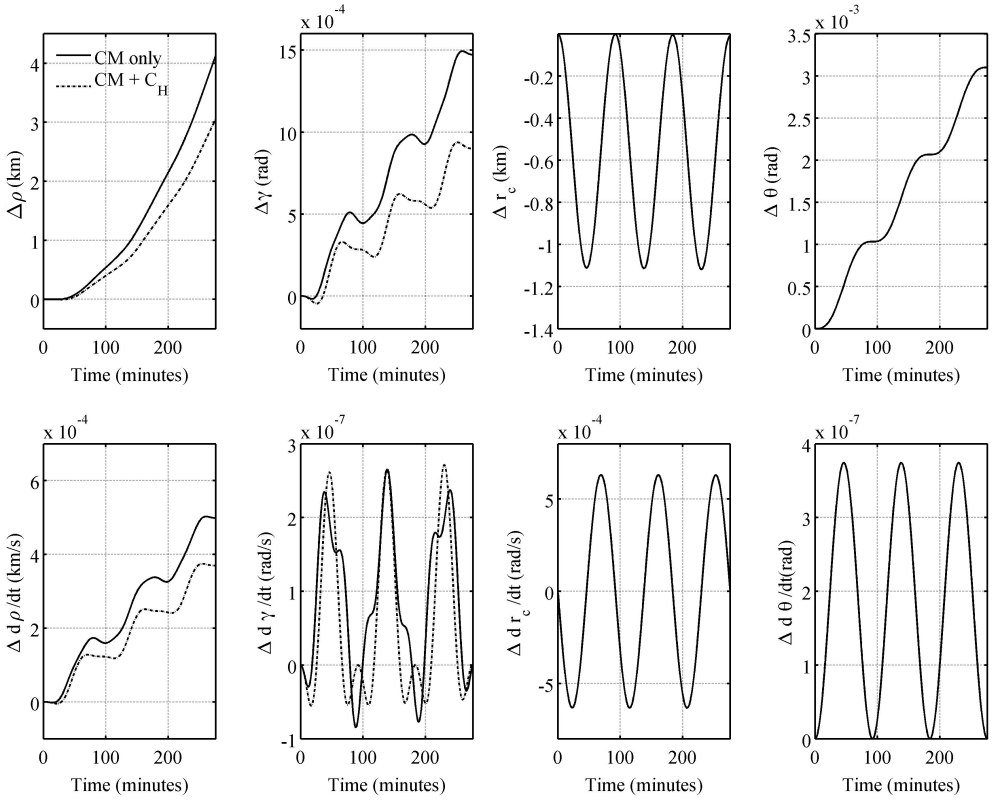

Figure 3. State errors due to simplifying the equations of motion by constraining the center of mass to follow a Keplerian orbit and conserve angular momentum.

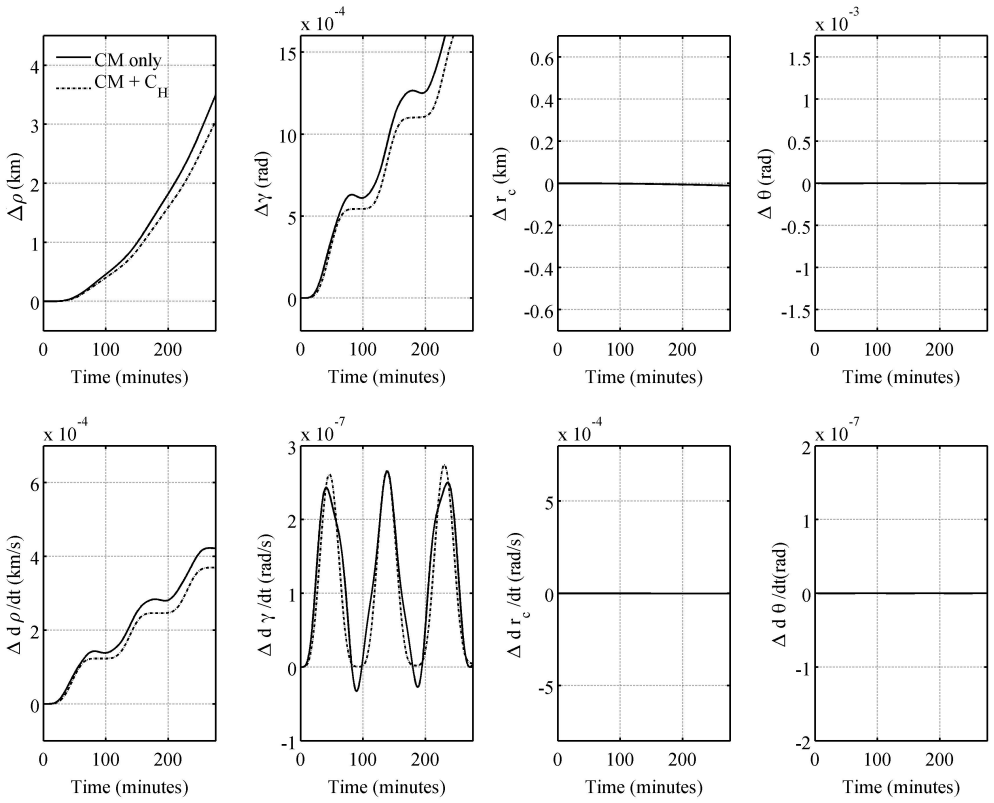

Figure 4. State errors due to simplifying the equations of motion by constraining the center of mass to follow its natural motion in the absence of PLP and conserve angular momentum.

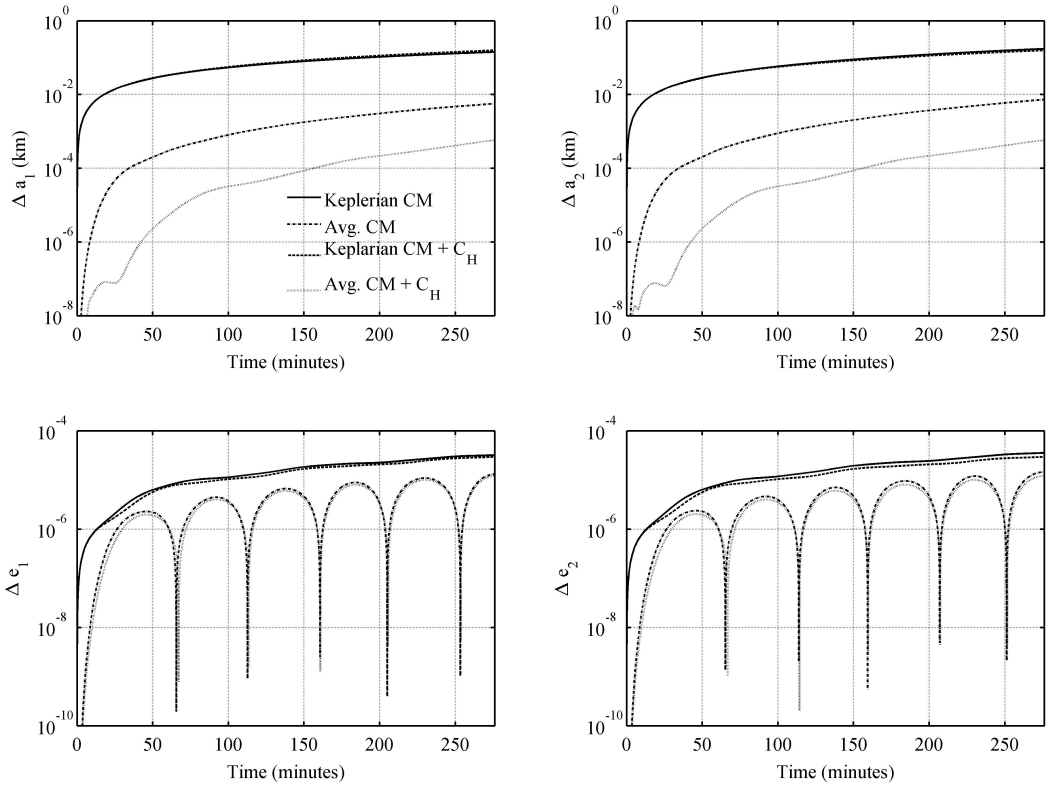

Figure 5. Errors in the semimajor axes a_1 and a_2 and eccentricities e_1 and e_2 due to the suite of assumptions regarding center of mass motion and angular momentum conservation.

the PLP input produces significantly lower errors in the evolution of the semimajor axes and eccentricities of the simulated system.

Figure 6 describes the true angular momentum and mechanical energy errors, ΔC_H and ΔC_E, and their linear approximations obtained from the simulation results of Figs. 3 and 4 in combination with Eq. (41). The errors observed in the integrals of motion could also be interpreted as how far the approximated motion drifts from the C_E surface in Fig. 2 given by Eq. (27) and the hyperdimensional surface defined by Eq. (30).

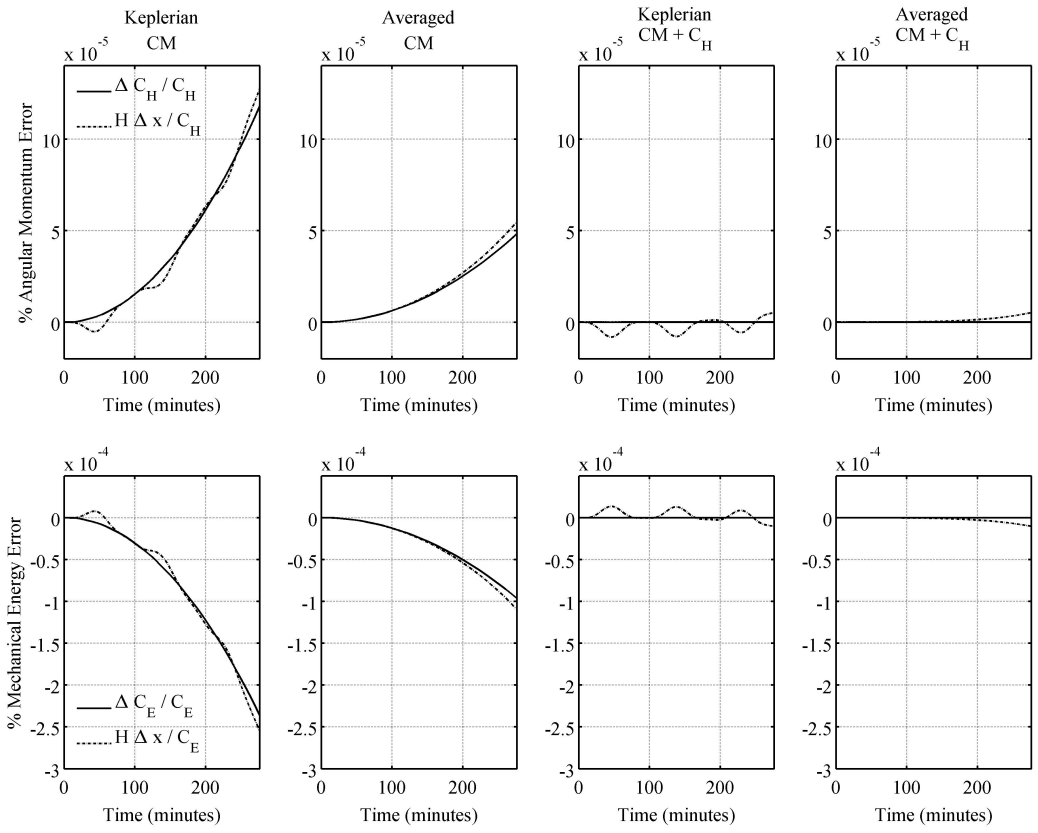

Figure 6. Comparing the true error of the integrals of motion, ΔC_E and ΔC_H, of the four simplified systems to the linear approximation of the error given by (41)

A close correlation between the time histories of the true and linear approximation of the error in these integrals of motion would indicate a valid linearizing approximation about the state of the constrained system. By conserving the angular momentum of the system, the system state is forced to conform to Eq. (25), and the corresponding combinations of orbital elements of the vehicles must also follow the relation described by Eq. (30). If similar constraints were imposed to conserve total energy, the same statements could be made regarding the system state and orbital elements with respect to Eqs. (23) and (27).

VI. Conclusion

While the differential equations describing the motion of a two-vehicle formation utilizing a conservative, internal force in orbit about a common central body are yet to be solved for analytically, integrals of motion exist for this system that bound the state trajectory. Two specific integrals of motion corresponding to mechanical energy and angular momentum are identified. As these time-invariant quantities remain conserved regardless of the choice of generalized coordinates, they also describe relationships among a subset of the osculating orbital elements of the vehicles.

In addition to relating system states at two distinct times, these conservation laws are also potentially useful in evaluating formations under certain simplifying assumptions. Enforcing the integrals of motion to remain time-invariant allows for the continued use of their defining equations in the analysis of the simplified system state. Furthermore, the error of these integrals of motion due to constraint forces are exactly known for a given state and can be approximated by a linear relationship to the state error between the true and simplified system. Furthermore, the induced error in the integrals of motion can be interpreted as the distance to a hyperdimensional surface described by time-varying orbital elements. While these two integrals of motion described cannot completely determine the error in the individual states, they define mathematical relationships to which the error states must adhere.

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